Uniform Sets and Complexity (The 8th Workshop on Stochastic Numerics)

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Uniform Sets and Complexity

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An element $\omega \in \{0,1\}^{\mathbb{N}}$ is called an *infinite* 0-1-*word* which is a mapping from $\mathbb{N}$ to $\{0,1\}$, while it is also considered as an infinite sequence $\omega(0)\omega(1)\omega(2)\cdots$ of 0 and 1. On the other hand, an element u in $\{0,1\}^* := \cup_{k=0}^{\infty}\{0,1\}^k$ is called a *finite* 0-1-*word* and represented as a finite sequence $u_1u_2\cdots u_k$ of 0 and 1, where k is such that $u \in \{0,1\}^k$, which is called the *length* of u and is denoted by $|u|$. We also denote $\{0,1\}^+ = \cup_{k=1}^{\infty}\{0,1\}^k$.

The *concatenation* uv or $u\omega$ of $u \in \{0,1\}^*$ with $v \in \{0,1\}^*$ or $\omega \in \{0,1\}^{\mathbb{N}}$ is defined as the word $u_1u_2\cdots u_kv_1v_2\cdots v_l$ or $u_1u_2\cdots u_k\omega(0)\omega(1)\omega(2)\cdots$, where $k=|u|$ and $l=|v|$, respectively. In this case, u is called a *prefix* of uv or $u\omega$, or equivalently, uv or $u\omega$ is called an *extension* of u.

For $u \in \{0,1\}^*$, the *cylinder set* $[u]$ determined by u is defined by

$$[u] = \{\omega \in \{0,1\}^{\mathbb{N}};\ u \text{ is a prefix of } \omega\}.$$

The *prefix tree* $G(\Omega) = (V,E)$ of a nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is defined to be a directed graph such that the set V of vertices is the set of cylinder sets $[u]$ which meet Ω, and the set E of edges is the set of the ordered pairs $([u],[v]) \in V \times V$ such that v is an immediate extension of u, that is, u is the prefix of v such that $|v| = |u|+1$.

Two nonempty closed sets $\Omega,\ \Lambda \subset \{0,1\}^{\mathbb{N}}$ are said to be *isomorphic* to each other if their prefix trees are isomorphic to each other. The class of all closed subsets of $\{0,1\}^{\mathbb{N}}$ isomorphic to Ω is denoted by $[\Omega]$ and is called the *language structure* of (or determined by) Ω.

Define

$$\Theta_0 := \{0^\infty\}\ ,\ \ \Theta_1 := \{1^\infty\},$$

$$\Theta_\delta := \{\omega \in \{0,1\}^{\mathbb{N}};\ \sum_{n\in\mathbb{N}}\omega(n) \le 1\},$$

$$\Theta_{1-\delta} := \{\omega \in \{0,1\}^{\mathbb{N}};\ \sum_{n\in\mathbb{N}}(1-\omega(n)) \le 1\},$$

$$\Theta_+ := \{\omega \in \{0,1\}^{\mathbb{N}};\ \omega \text{ is increasing}\},$$

$$\Theta_- := \{\omega \in \{0,1\}^{\mathbb{N}};\ \omega \text{ is decreasing}\},$$

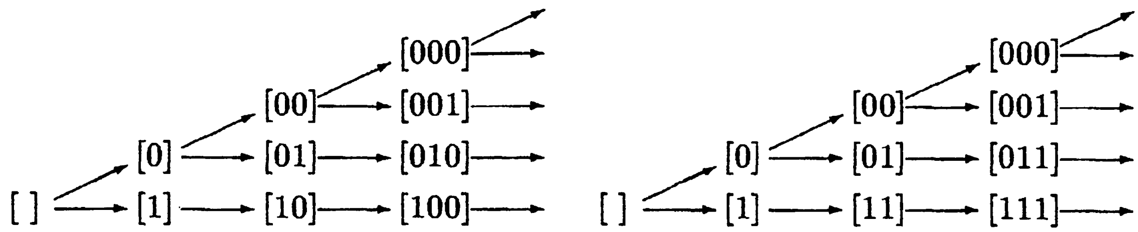

Figure 1: $G(\Theta_\delta)$ (left) and $G(\Theta_+)$ (right)

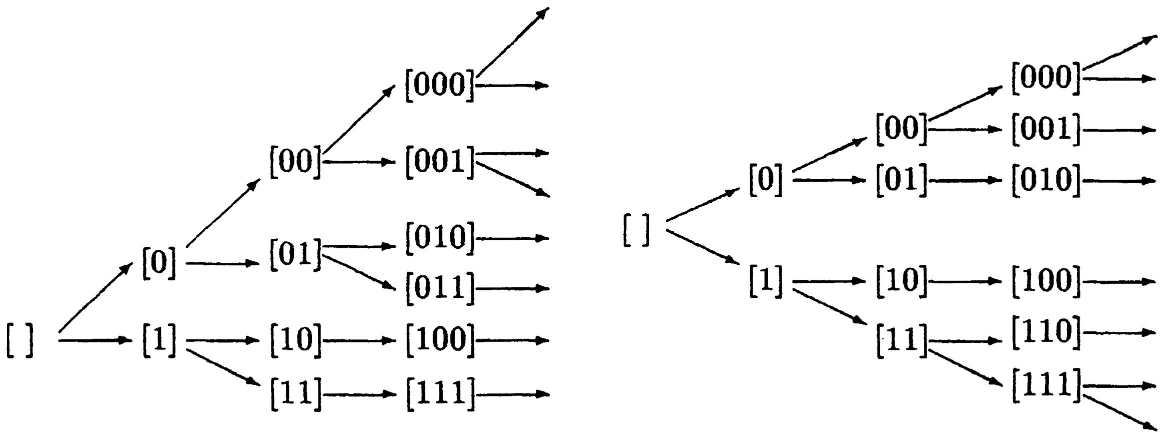

Figure 2: $G(\Theta_\delta \cup \Theta_+)$ (left) and $G(\Theta_\delta \cup \Theta_-)$ (right)

where $a^\infty = aaa\cdots$ for $a \in \{0,1\}$ and $\omega \in \{0,1\}^{\mathbf{N}}$ is called *increasing* (*decreasing*) if $\omega(n) \leq \omega(m)$ ($\omega(n) \geq \omega(m)$, respectively) for any $n < m$.

All of Θ_δ, $\Theta_{1-\delta}$, Θ_+, Θ_- are isomorphic to each other since for example, $G(\Theta_\delta)$ and $G(\Theta_+)$ are isomorphic (Figure 1). It also holds that $\Theta_\delta \cup \Theta_-$ and $\Theta_+ \cup \Theta_-$ are isomorphic, while $\Theta_\delta \cup \Theta_+$ is not isomorphic to $\Theta_\delta \cup \Theta_-$ (Figure 2).

Definition 1. For a nonempty closed set $\Omega \subset \{0,1\}^\Sigma$, define the *complexity function* $p_\Omega(S) := \#\pi_S\Omega$, which is a function of finite sets $S \subset \Sigma$, where $\#$ denotes the number of elements in a set and $\pi_S : \{0,1\}^\Sigma \to \{0,1\}^S$ is the projection. We call Ω a *uniform set* if $p_\Omega(S)$ depends only on $\#S$. In this case, the function $p_\Omega(k) := p_\Omega(S)$ of $k = 1, 2, \cdots$, where $\#S = k$, is called the *uniform complexity function* of Ω. We also define the *maximal pattern complexity function* of Ω as $p^*_\Omega(k) := \sup_{S;\#S=k} p_\Omega(S)$ $(k = 1, 2, \cdots)$. Note that $p_\Omega(k) = p^*_\Omega(k)$ $(k = 1, 2, \cdots)$ if Ω is a uniform set.

Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ be an infinite subset of $\mathbf{N}$. For $\omega \in$

$\{0,1\}^{\mathbb{N}}$ and $\Omega \subset \{0,1\}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \{0,1\}^{\mathbb{N}}$ by

$$\begin{aligned} \omega[\mathcal{N}](n) &:= \omega(N_n) \quad (n \in \mathbb{N}) \\ \Omega[\mathcal{N}] &:= \{\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}. \end{aligned}$$

We sometimes identify the infinite subset $\mathcal{N} \subset \mathbb{N}$ with an increasing injection $n \mapsto N_n$ from $\mathbb{N}$ into itself. We use the same notation $\Omega[S]$ for a finite set $S = \{s_1 < s_2 < \cdots < s_k\} \subset \mathbb{N}$ in the sense that

$$\Omega[S] = \{\omega(s_1)\omega(s_2)\cdots\omega(s_k) \in \{0,1\}^k;\ \omega \in \Omega\}.$$

For $\Omega \subset \{0,1\}^{\Sigma}$, where Σ is a countably infinite set, and an injection $\psi : \mathbb{N} \to \Sigma$, denote

$$\Omega \circ \psi := \{\omega \circ \psi \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}.$$

Note that if Ω is a uniform set, then $\Omega \circ \psi$ is also a uniform set with the same complexity function.

Definition 2. A nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is called a *super-stationary* set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$. Note that a super-stationary set is a uniform set and all of Θ_0, Θ_1, Θ_δ, $\Theta_{1-\delta}$, Θ_+, Θ_- are super-stationary sets.

Definition 3. A nonempty closed set $\Omega \subset \{0,1\}^{\Sigma}$ is said to have a *primitive factor* $[\Omega\circ\phi]$ if $\Omega\circ\phi$ is a super-stationary set, where $\phi : \mathbb{N} \to \Sigma$ is an injection and $[\Omega \circ \phi]$ is the language structure determined by $\Omega \circ \phi$.

Definition 4. Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be a nonempty closed set. For $\omega \in \Omega$ and $k \in \mathbb{N}$, we denote $\omega|_k = \omega(0)\omega(1)\cdots\omega(k-1) \in \{0,1\}^k$. Let Ω' be the set of accumulating points of Ω, that is,

$$\Omega' = \{\omega \in \Omega;\ \#([\omega|_k] \cap \Omega) = \infty \text{ for any } k \in \mathbb{N}\}.$$

We call Ω' the *derived set* of Ω. Clearly, Ω' is a closed set (possibly, the empty set). We denote $\Omega^{(0)} = \Omega$ and $\Omega^{(i)} = (\Omega^{(i-1)})'$ for $i = 1, 2, \cdots$. The *degree* of Ω is defined to be $d = 0, 1, 2, \cdots$ such that $\Omega^{(d)} \neq \emptyset$ and $\Omega^{(d+1)} = \emptyset$, if such d exists, otherwise, ∞. The degree of Ω is denoted by $\deg \Omega$. For completeness, we define $\emptyset' = \emptyset$ and $\deg \emptyset = -1$.

We have the following results.

Theorem 5. (Kamae [1]) *Let Ω be a nonempty closed subset of $\{0,1\}^{\Sigma}$, where Σ is a countably infinite set.*
(1) *If there exists an injection $\psi : \mathbb{N} \to \Sigma$ such that $\deg(\Omega \circ \rho) < \infty$, then there exists an increasing injection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\Omega \circ \phi$ is a super-stationary set.*
(2) *If $\deg(\Omega \circ \rho) = \infty$ for any injection $\rho : \mathbb{N} \to \Sigma$, then $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$.*
Hence, any uniform set has a primitive factor and any uniform complexity function is realized by a super-stationary set.

Remark 6. (1) of the Main Theorem can be generalized easily to the case of general finite alphabet.

For $\xi = \xi_1\xi_2\cdots\xi_k \in \{0,1\}^k$ and $\eta = \eta_1\eta_2\cdots\eta_l \in \{0,1\}^l$ with $k \le l$, we say that ξ is a *super-subword* of η, if $\xi = \eta_{s_1}\eta_{s_2}\cdots\eta_{s_k}$ holds for some $1 \le s_1 < s_2 < \cdots < s_k \le l$. For this ξ and $\omega \in \{0,1\}^{\mathbb{N}}$, we say that ξ is a *super-subword* of ω, if $\xi = \omega(s_1)\omega(s_2)\cdots\omega(s_k)$ holds for some $0 \le s_1 < s_2 < \cdots < s_k < \infty$. In these cases, we denote $\xi \ll \eta$ or $\xi \ll \omega$.

For $\xi \in \{0,1\}^*$, denote

$$\mathcal{P}(\xi) := \{\omega \in \{0,1\}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\},$$

that is, $\mathcal{P}(\xi)$ is the set of infinite 0-1-words with the prohibited word ξ as its super-subword. Denote for $\Xi \subset \{0,1\}^*$,

$$\mathcal{Q}(\Xi) := \bigcup_{\xi\in\Xi} \mathcal{P}(\xi) \text{ and } \mathcal{P}(\Xi) := \bigcap_{\xi\in\Xi} \mathcal{P}(\xi).$$

We call $\eta \in \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ a *cover* of Ξ if $\xi \ll \eta$ holds for any $\xi \in \Xi$. It is called a *minimal cover* if in addition, any $\zeta \underset{\neq}{\ll} \eta$ is not a cover of Ξ. Let $\mathbb{L}(\Xi)$ be the set of minimal covers of Ξ.

Theorem 7. (T. Kamae, H. Rao, B. Tan and Y-M. Xue [2]) (1) *The class of super-stationary sets other than $\{0,1\}^{\mathbb{N}}$ coincides with the class of sets $\mathcal{Q}(\Xi)$ with nonempty finite sets $\Xi \subset \{0,1\}^+$. It also coincides with the class of sets $\mathcal{P}(\mathbb{L}(\Xi))$ with nonempty finite sets $\Xi \subset \{0,1\}^+$.*
(2) *The complexity function $p_\Omega(k)$ of a super-stationary set Ω other than $\{0,1\}^{\mathbb{N}}$ is a polynomial function of k for large k.*

The following Corollary follows from abov 2 threorems.

Corollary 8. *The complexity function $p_\Omega(k)$ of a uniform set Ω is either 2^k $(k = 1,2,\cdots)$ or a polynomial function of k for large k.*

Let X be a metrizable space with a continuous group or semi-group action G. For a family of subsets $A_1, A_2, \cdots, A_k$ of X, let $\mathbb{P}(\{A_i; i = 1,2,\cdots,k\})$ denote the *partition* of X generated by these subsets, that is, the family of nonempty sets of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_k^{i_k} \quad (i_1, i_2, \cdots, i_k \in \{0,1\}),$$

where for a set $A \subset X$, we denote $A^1 = A$ and $A^0 = X \setminus A$.

Let D be a nonempty subset of X. Define the *maximal pattern complexity* function $p^*_{X,G,D}$ of the triple (X,G,D) by

$$p^*_{X,G,D}(k) = \sup_{\tau\subset G,\ \#\tau=k} \#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) \quad (k = 1,2,\cdots). \tag{1}$$

Definition 9. For a set U and $k \in \mathbb{N}$, $\mathcal{F}_k(U)$ denotes the family of sets $S \subset U$ with $\#S = k$. A countably infinite subset Σ of G is called an *optimal position* of the triple (X, G, D) if

$$\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{X,G,D}(k), \tag{2}$$

holds for any $k = 1, 2, \cdots$ and $\tau \in \mathcal{F}_k(\Sigma)$.

Let $\Sigma \subset G$ be a countably infinite set. We call $\omega \in \{0,1\}^\Sigma$ a *name* of the partition $\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \Sigma\})$ if there exists $x \in X$ such that

$$\omega(\sigma) = \begin{cases} 1 & x \in \sigma^{-1}D \\ 0 & x \notin \sigma^{-1}D. \end{cases} \tag{3}$$

The closure of the set of names of the partition $\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \Sigma\})$ is called the *name set* of Σ with respect to the triple (X, G, D).

The following theorem is clear from the definitions.

Theorem 10. *The name set of any optimal position Σ of a triple (X, G, D) is a uniform set with the uniform complexity function $p^*_{X,G,D}$.*

Example 11. Let $X = G = \mathbb{R}/\mathbb{Z}$. The action of $g \in G$ maps $x \in X$ to $x+g \in X$. Let D be an interval $[a, b)$ in X such that $a < b < a+1$. Then, we have $p^*_{D,G}(k) = 2k$ $(k = 1, 2, \cdots)$. In this case, a countably infinite subset Σ of G is an optimal position of (X, G, D) if and only if for any $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, $D - \sigma$ and $D - \sigma'$ intersect as well as their complements. This is also equivalent to that $\#\mathbb{P}(\{\sigma^{-1}D,\ \sigma'^{-1}D\}) = 4$ for any $\{\sigma, \sigma'\} \in \mathcal{F}_2(\Sigma)$.

Let Ω be the name set of an optimal position Σ. Then, Ω is known to have the unique primitive factor $[\Theta_\delta \cup \Theta_-] = [\mathcal{Q}(11, 01)]$ ([?]).

Example 12. Let $X = \mathbb{R}^2$ and $G = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^2$. The action of $(\theta, (u, v))$ in G maps $(x, y) \in X$ to the following $(x', y') \in X$:

$$\begin{cases} x' = x\cos\theta - y\sin\theta + u \\ y' = x\sin\theta + y\cos\theta + v. \end{cases}$$

Let D be a line in X. Then, $g^{-1}D$ is also a line for any $g \in G$ and we have $p^*_{X,G,D}(k) = (1/2)k^2 + (1/2)k + 1$ $(k = 1, 2, \cdots)$. In this case, Σ is an optimal position if and only if Σ is a countably infinite subset of G such that
(1) for any $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, $\sigma^{-1}D \cap \sigma'^{-1}D \neq \emptyset$, and
(2) for any $\sigma, \sigma', \sigma'' \in \Sigma$ which are different each other,

$$\sigma^{-1}D \cap \sigma'^{-1}D \cap \sigma''^{-1}D = \emptyset.$$

Let Ω be the name set of an optimal position Σ. Then,

$$\Omega = \{\omega \in \{0,1\}^\Sigma;\ \sum_{\sigma\in\Sigma} \omega(\sigma) \leq 2\}.$$

Hence, Ω has the unique primitive factor $[\mathcal{Q}(111)]$.

Example 13. (Y-M. Xue [5]) Let $X = G = \mathbb{R}^2$. The action of $g = (g_1, g_2) \in G$ maps $(x, y) \in X$ to $(x+g_1, y+g_2) \in X$. Let $D := \{(x,y) \in \mathbb{R}^2;\ x^2+y^2 \leq 1\}$ be the unit disk. Then, we have $p^*_{X,G,D}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. In this case, a countably infinite subset Σ of G is an optimal position if and only if $\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{X,G,D}(3) = 8$ for any $\tau \in \mathcal{F}_3(\Sigma)$. Moreover, Σ satisfies this condition if $\Sigma \subset \{g \in G;\ g_1^2 + g_2^2 = r^2\}$ with $0 < r < 1$. Moreover, the name set Ω has a unique primitive factor $[\mathcal{Q}(101, 010)]$.

All the examples so far admit a finitely determined optimal position. The following example does not admit an optimal position.

Example 14. Let $X = \mathbf{T}_1 \cup \mathbf{T}_2$ and $G = \mathbf{T}_1 \times \mathbf{T}_2$, where $\mathbf{T}_i \cong \mathbb{R}/\mathbb{Z}$ $(i = 1, 2)$ and $\mathbf{T}_1, \mathbf{T}_2$ are disjoint each other. The action of $g = (g_1, g_2) \in G$ maps $x \in \mathbf{T}_i$ to $x + g_i \in \mathbf{T}_i$ for $i = 1, 2$. Let $D = [a_1, b_1) \cup [a_2, b_2)$, where $[a_i, b_i) \subset \mathbf{T}_i$ and $a_i < b_i < a_i + 1$ for $i = 1, 2$.

Then, we have $p^*_{X,G,D}(k) = 4k - 4$ $(k = 2, 3, \cdots)$. In this case, there is no optimal position since for any infinite subset Σ of G, there exists a sequence $g_n = (g_{n,1}, g_{n,2}) \in \Sigma$ for $n = 1, 2, \cdots$ such that $g_{n,i}$ converges monotonously to, say $c_i \in \mathbf{T}_i$, for $i = 1, 2$. Then, for any sufficiently large n_0, $\#\mathbb{P}(\{g_n^{-1}D;\ n = n_0 + 1, n_0 + 2, n_0 + 3\}) = 6$ but not 8.

Definition 15. A nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is called a *stationary* set if $T\Omega = \Omega$, where $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the shift. Note that a super-stationary set is always stationary since $T\Omega = \Omega[\{1, 2, \cdots\}]$. We call $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$ an *optimal window* of Ω if $p_\Omega(S) = p^*_\Omega(k)$ for any $k = 1, 2, \cdots$ and $S \subset \mathcal{N}$ with $\#S = k$.

Take a stationary set $\Omega \subset \{0,1\}^{\mathbb{N}}$ as X and the additive semi-group $\mathbb{N}$ as G. Let the action of $n \in \mathbb{N}$ to $\omega \in \Omega$ be $T^n\omega$. Let $D = \{\omega \in \Omega;\ \omega(0) = 1\}$. In this case, it is easy to see that

Theorem 16. *For an infinite subset $\mathcal{N}$ of $\mathbb{N}$, $\mathcal{N}$ is an optimal position of $(\Omega, \mathbb{N}, D)$ if and only if $\mathcal{N}$ is an optimal window of Ω.*

Hence, the following theorem follows from Theorem 4.1 of T. Kamae, H. Rao, B. Tan, Y-M. Xue [3].

Theorem 17. *Let $\alpha \in \{0,1\}^{\mathbb{N}}$ be a recurrent pattern Sturmian word. Let $X = \overline{O}(\alpha)$, $G = \{T^n;\ n \in \mathbb{N}\}$ and $D = \{\omega \in \Omega;\ \omega(0) = 1\}$. Then, an optimal position of the triple (X, G, D) exists.*

Example 18. Let $\Omega = \overline{O}(\alpha)$ with the non-simple Toeplitz word $\alpha \in \{0,1\}^{\mathbb{N}}$ defined in Example 3 in N. Gjini, T. Kamae, B. Tan, and Y.-M. Xue [4]. Then, $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$ holds. In this case, an optimal window does not exist. Take an arbitrary $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$. For any $k \in \mathbb{N}$, there exists $K \in \mathbb{N}$ with $K \geq k$ and $\xi \in \{0,1\}^K$ such that $\alpha = (\xi a_0)(\xi a_1)(\xi a_2) \cdots$ holds with $a_0, a_1, a_2 \cdots \in \{0, 1\}$. There exists such

K with the property that there exist 3 elements in $\mathcal{N}$, say $N_u < N_v < N_w$ with $N_u \not\equiv N_v \equiv N_w$ modulo $K+1$. Then, either 001 or 101 is not in $\Omega[\{N_u, N_v, N_w\}]$. Hence, $\mathcal{N}$ is not an optimal window.

The following is the list of the language structures and the complexity functions with degree ≤ 1.

(1) $[\Theta_0] = [\mathcal{Q}(1)]$, $p_\Omega(k) = 1$,
(2) $[\Theta_0 \cup \Theta_1] = [\mathcal{Q}(0,1)]$, $p_\Omega(k) = 2$,
(3) $[\Theta_\delta] = [\mathcal{Q}(11)]$, $p_\Omega(k) = k+1$,
(4) $[\Theta_\delta \cup \Theta_1] = [\mathcal{Q}(11,0)]$, $p_\Omega(k) = k+2-1_{k=1}$,
(5) $[\Theta_\delta \cup \Theta_+] = [\mathcal{Q}(11,10)]$, $p_\Omega(k) = 2k$,
(6) $[\Theta_\delta \cup \Theta_-] = [\mathcal{Q}(11,01)]$, $p_\Omega(k) = 2k$,
(7) $[\Theta_\delta \cup \Theta_{1-\delta}] = [\mathcal{Q}(11,00)]$, $p_\Omega(k) = 2k+2-2\cdot 1_{k=1}$,
(8) $[\Theta_\delta \cup \Theta_+ \cup \Theta_-] = [\mathcal{Q}(11,10,01)]$, $p_\Omega(k) = 3k-2+1_{k=1}$,
(9) $[\Theta_\delta \cup \Theta_{1-\delta} \cup \Theta_+] = [\mathcal{Q}(11,00,10)]$, $p_\Omega(k) = 3k-1-1_{k=2}$,
(10) $[\Theta_\delta \cup \Theta_{1-\delta} \cup \Theta_+ \cup \Theta_-] = [\mathcal{Q}(11,00,10,01)]$, $p_\Omega(k) = 4k-4+2\cdot 1_{k=1}$.

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